Journal of Applied Mathematics and Mechanics 72 (2008) 338–346

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The solution of three-dimensional friction contact problems☆

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ARTICLE INFO

Article history:
Received 19 June 2007

ABSTRACT

A variational method is developed for solving friction contact problems, in which the friction obeys Coulomb's of friction law in velocities, and numerical solutions of three-dimensional problems of the contact of a sphere, a cylinder of finite length and a cube with an elastic half-space are constructed. It is established that the maximum frictional forces correspond to a boundary point of the regions of adhesion and slippage. When the number of steps,increase this maximum decreases, and the distribution of the frictional forces becomes smoother. Certain undesirable effects that can arise during numerical implementation of the method – numerical artefacts – are described. These effects can occur in the numerical solution of problems with a different physical content, the mathematical structure of which is similar to the structure of the contact problems investigated, as the artefacts are caused by the presence of unilateral constraints and by the dependence on external effects of the region in which unilateral constraints with an equally sign occur. This problem is solved by an appropriate choice of the load-step zero approximations.

1. Introduction

The problem of the contact of a moving rigid rough punch with an elastic deformable body is considered. The shapes of the punch and the body can be arbitrary, and the contact area and the boundary of the regions of adhesion and slip page are not known in advance and must be determined in the course of the solution. The Amonton–Coulomb friction law contains the velocities of relative displacements, and the problem must therefore be formulated and solved in time increments or in relation to increments in the external actions in the case of the quasi-static problems considered in this paper. As in earlier research,[1–4] the variational method is used, enabling us to carry out a theoretical study and to formulate an iterational method of solution. Each load-step iteration consists of solving a sequence of problems of the minimization of a certain functional, and the latter, in turn, reduces to solving normal (non-contact) problems of elasticity theory. In the numerical implementation, discretization with respect to the spatial variables is carried out by the boundary element method, and infiniterimal increments and loads and the solutions are replaced by finite differences.

The friction contact problem of the penetration of a flat rectangular punch into an elastic half-plane for an unknown boundary of the regions of adhesion and slip-page was first solved by Galin.[5] The velocities in the friction law were replaced by displacements, and it was established that the boundary mentioned is independent of the clamping force and depends only on the coefficient of friction f and Poisson's ratio ν. The coordinates of the boundary were found numerically for ten values of the coefficient of friction.

Subsequently, an equation was found for the boundary[6] that was in good agreement with Galin's solution.

A solution was found[6,7] of the axisymmetric friction contact problem with an unknown boundary line of the regions of adhesion and slip page. A review of results obtained in recent years is available, and the solutions of certain new contact problems with friction, wear and adhesion have been constructed.[8]

In all the cited and majority published studies on this problem, the following constraining assumptions are used:

(1) the friction law relates the friction forces to the normal pressure and relative displacements in the contact area;
(2) the normal pressure is assumed to be known and to be independent of the friction forces;

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 3, pp. 485–496, 2008.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2008.07.014

(3) the direction of the vector of the frictional forces is independent of the change in shape of the contacting bodies owing to their strain; for example, in the problem of the contact of a rigid punch with a half-space, the friction forces are parallel to the undeformed boundary of the half-space.

The procedure used below, free of the constraints indicated, was developed earlier.[1–4]

2. Basic notation and relations, and local formulation

Let the deformable body in contact with an absolutely rigid, rough, moving punch occupy an arbitrary region Ω with boundary $\sum = \partial\Omega$.

We will use the following notation: $\hat{\sigma}$ is the stress tensor (the circumflex denotes second rank), $\hat{\varepsilon} = (\nabla u + \nabla u^T)/2$ is the Cauchy tensor of small strains in a Cartesian system: $\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, $\mathbf{u}$ is the displacement vector of , $\mathbf{v}$ is the vector of the unit outword normal to the boundary $\sum$, $\sigma = \hat{\sigma} \cdot v$ is the vector of forces at boundary Σ, $\sigma = \sigma_N v + \sigma_T$ is the expansion of the vector of surface forces in tangential (to the boundary) and normal components and $\mathbf{x}$ is the radius vector of a point.

Consider the process of change in the stress–strain state of the deformable body as a function of the process of change in external effects. The process parameter will be denoted by t; in dynamics, t is time, but in the problems considered here it is the kinematic parameter defining the position of the punch (for example, the length of the path of translational motion) for which rotation is possible, which is also depends on the path along the trajectory.

The local formulation of the problem contains equations at points of the region Ω and equations and inequalities on the boundary $\sum$. We will assume that the mass forces are zero; then, in region Ω, we have:

The equations of equilibrium

$$\nabla \cdot \hat{\boldsymbol{\sigma}} = 0 \tag{2.1}$$

Hooke's law

$$\hat{\boldsymbol{\sigma}} = \hat{\hat{a}} \cdot\cdot\, \hat{\boldsymbol{\varepsilon}} \tag{2.2}$$

and Cauchy's formulae given above for the components of the tensor of small strains; $\hat{\hat{a}}$ is the modulus of elasticity tensor.

It is assumed that $\Sigma = \Sigma_\sigma \cup \Sigma_u \cup \Sigma_C$. On the part of the boundary $\sum_\sigma \subset \sum$, surface forces with density $\mathbf{P}$ are specified:

$$\hat{\boldsymbol{\sigma}}\mathbf{v}|_{\Sigma_\sigma} = \mathbf{P}(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma_\sigma \tag{2.3}$$

and on the part of the boundary $\sum_u \subset \sum$ the following displacements are specified:

$$\mathbf{u}|_{\Sigma_u} = \mathbf{U}(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma_u \tag{2.4}$$

where $P(x,t)$ and $U(x,t)$ are specified functions. The points of the surface $\sum_c$ can enter into contact with the punch.

To formulate the boundary conditions on $\sum_c$, we will introduce two systems of coordinates: the stationary system $Oxyz \equiv Ox_1x_2x_3$ and the system $O_1\xi_1\xi_2\xi_3$ connected with the moving punch; for simplicity, we will assume that both systems are Cartesian and that at the initial instant of time they coincide. Let the surface of the punch at the instant when loading stats be described by the equation

$$\Psi(\boldsymbol{\xi}) = 0 \tag{2.5}$$

We will assume that $\Psi(\xi) > 0$ for points with coordinates ξ outside the region occupied by the punch, and $\Psi(\xi) < 0$ for points within the punch. We will specify the motion of the punch by the vector U_p of translational motion and by the matrix of rotation $\hat{A}$, the elements of which are expressed by the Euler angles between the axes of the moving and stationary systems of coordinates. Note that, to formulate and solve friction contact problems, it is necessary to specify infinitesimal displacements and rotations, for which the law of summation holds. Problems in which only transnational motion of the punch is specified or in which firstly the translational movement of the die and then its rotation are specified successively are examined. The law of summation will then hold for the cpmplete motions and rotations, i.e.

$$\mathbf{x} = \mathbf{U}_p + \hat{A} \cdot \boldsymbol{\xi} \tag{2.6}$$

The case of arbitrary motion of the punch was examined earlier.[9]

Considering the ratio (2.6) as an equation in the variables ξ, we find that

$$\boldsymbol{\xi} = \hat{A}^{-1} \cdot (\mathbf{x} - \mathbf{U}_p) \tag{2.7}$$

As a result of strain, the arbitrary point $x \in \sum_c$ will occupy the position $x + u(x,t)$; then formula (2.7) and the assumptions formulated above with regard to the function Ψ enable us to write the inequality

$$\boldsymbol{\Psi}[\hat{A}^{-1} \cdot (\mathbf{x} + \mathbf{u}(\mathbf{x}, t) - \mathbf{U}_p)] \geq 0, \quad \forall \mathbf{x} \in \Sigma_C \tag{2.8}$$

the significance of which lies in the fact that points of the boundary of the deformable body are unable to penetrate into the punch.

We will assume that the normal forces are compressive; then

$$\boldsymbol{\sigma}_N(\mathbf{x}, t) \leq 0, \quad \forall \mathbf{x} \in \Sigma_C \tag{2.9}$$

The following equation then holds

$$\Psi[\hat{A}^{-1} \cdot (\mathbf{x} + \mathbf{u}(\mathbf{x}, t) + \mathbf{U}_p)]\boldsymbol{\sigma}_N(\mathbf{x}, t) = 0, \ \forall \mathbf{x} \in \Sigma_C \tag{2.10}$$

the significance of which lies in the fact that either there is contact at each point of the boundary or, presumably, there is zero pressure on the surface.

For closure of the model it is necessary to specify the ratios for the tangential components of the surface forces in the contact area. when there is no friction, these components are equal to zero:

$$\boldsymbol{\sigma}_T(\mathbf{x}, t) = 0, \forall \mathbf{x} \in \Sigma_C \tag{2.11}$$

If, however, Coulomb's law of dry friction applies in the contact area, then

$$|\boldsymbol{\sigma}_T| \le f|\sigma_N| \Rightarrow \dot{\mathbf{u}}_T = 0 \tag{2.12}$$

$$|\boldsymbol{\sigma}_T| \le f|\sigma_N| \Rightarrow \exists \kappa \ge 0 : \dot{\mathbf{u}}_T = -\kappa \boldsymbol{\sigma}_T \tag{2.13}$$

If slippage occurs, then $|\dot{u}_T| > 0$ and

$$\boldsymbol{\sigma}_T / |\boldsymbol{\sigma}_T| = -\dot{\mathbf{u}}_T / |\dot{\mathbf{u}}_T| \tag{2.14}$$

The quantity $|\dot{u}_T|$ is equal to the relative slip velocity of points of the body with respect to the punch, i.e. to the derivative of the tangential component of the vector of relative displacements with respect to the parameter t.

The local formulation of the problem consists in finding the displacement field u(x,t), $x \in \Omega$, $t \in$[0,T] from relations (2.1)–(2.4) and (2.8–2.13).

Remark. The problem of contact between two (or more) deformable bodies for the case of small strains is normally formulated and solved using Hertz hypotheses: the initial contact of the bodies occurs at a point, and the non-penetration condition is formulated for projections of the displacements onto the normal to the general tangential plane at the initial point of contact. A general theory for the final strains and displacements has been developed[10,11] using Riemannian geometry for the contact surfaces.

3. Variational formulation of the problem and method of solution

We will first consider the problem of taking into account the velocities, i.e. the dependences on the process history. We will introduce the difference (for brevity, we will omit the dependence on **x**)

$$\delta^* \mathbf{u}^t = \mathbf{u}(t + dt) - \mathbf{u}(t) \equiv \mathbf{u}^{t+dt} - \mathbf{u}^t \tag{3.1}$$

By of construction, the quantity u^t is known, and the displacement field u^{t+dt} is to be determined. We will use du^t to denote the linear part of this displacement, and for the variation we will retain the notation δ. Then

$$\delta^* \mathbf{u}^t = \mathbf{u}^{t+dt} - \mathbf{u}^t, \quad \delta \mathbf{u}^t = \mathbf{w} - \mathbf{u}^t, \quad d\mathbf{u}^t = \dot{\mathbf{u}}(\mathbf{x}, t)dt, \quad d\boldsymbol{\upsilon}^t = \dot{\boldsymbol{\upsilon}}(\mathbf{x}, t)dt \tag{3.2}$$

where **w** is the kinematically admissible field of displacements for the instant of time $t+dt$, $\dot{u}(t)$ is the field of true velocities at instant of time t and $\dot{\upsilon}(t)$ is the field of possible velocities.

The definition of the kinematic admissibility of displacements includes satisfaction of boundary condition (2.4) and non-penetration condition (2.8), and also continuity in the region $\bar{\Omega}$ with suitable constraints imposed on the smoothness in the region Ω.

The problem of determining the kinematically admissible velocities on the surface $\sum_c$ is solved by using Ostrogradskii's hypothesis: these velocities are arbitrary (naturally, they differ by an infinitesimal amount from the true velocities) at those points at which there is no contact, i.e. the non-penetration condition is satisfied with a sign of strict inequality; at points of contact, the possible velocities can be directed only outside the punch. The given hypothesis leads to the following definition of the set of kinematically admissible (possible) velocities

$$\dot{K}_u = \{\dot{\boldsymbol{\upsilon}} | \dot{\boldsymbol{\upsilon}} = \dot{\mathbf{u}} + \delta\dot{\mathbf{u}}, \Psi(\boldsymbol{\alpha})'_\alpha \cdot (A^{-1} \cdot \delta\dot{\mathbf{u}}) \ge 0, \forall \mathbf{x} \in \Sigma_C^t\} \tag{3.3}$$

where

$$\Sigma_C^t = \left\{ \mathbf{x} | \mathbf{x} \in \Sigma_C; \Psi(\boldsymbol{\alpha}(\mathbf{x})) = 0; \hat{A}' \cdot (\dot{\hat{A}}^{-1}(\mathbf{x} - \mathbf{U}_p + \mathbf{u}) + \hat{A}(-\dot{\mathbf{U}}_p + \dot{\mathbf{u}})) = 0 \right\}$$

$$\hat{A} = \hat{A}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} = \hat{A}^{-1}(\mathbf{x} - \mathbf{U}_p + \mathbf{u}) \tag{3.4}$$

The prime denotes a derivative with respect to the variable α.

Using the principle of possible velocities and the method developed earlier (see, for example, Ref. 2), we establish that any solution of the local problem formulated above satisfies the variational inequality

$$a(\mathbf{u}, \delta\dot{\mathbf{u}}) + \int_{\Sigma_c} f|\sigma_N(\mathbf{u})| \left|(|\dot{\boldsymbol{\upsilon}}_T| - |\dot{\mathbf{u}}_T|)\right| d\Sigma \ge L(\delta\dot{\mathbf{u}}), \ \forall(\delta\dot{\mathbf{u}}) = \dot{\boldsymbol{\upsilon}} - \dot{\mathbf{u}}, \quad \dot{\boldsymbol{\upsilon}} \in \dot{K}_u \tag{3.5}$$

in which

$$a(\mathbf{u}, \delta\mathbf{u}) = \int_{\Omega} \hat{\sigma}(\mathbf{u}) \cdot\cdot\, \hat{\varepsilon}(\delta\mathbf{u}) d\Omega, \quad L(\delta\mathbf{u}) = \int_{\Sigma_\sigma} \mathbf{P} \cdot \delta\mathbf{u} d\Sigma \tag{3.6}$$

Any solution of inequality (3.5) will be a generalized solution of the local problem.

Inequality (3.5) can be classified as a quasi-variational inequality characterized by the fact that the constraints imposed on the solutions themselves depend on the solution. This term (for abstract formulation) was first introduced by Tartar.[12] To solve inequality (3.5), the local potential method[2] can be used. This leads to doubling of the number of functions sought and for this reason, obviously it, has not been used in practice.

For numerical implementation it is more convenient to change to a problem in which only displacements occur. To this end, in inequality (3.5) we will replace the velocities using to formulae (3.2); assuming that $w = \dot{\upsilon} dt$, we arrive at a variational inequality for the field $\mathrm{u}^{t+dt} \equiv \mathrm{u}$

$$a(\mathbf{u}, \boldsymbol{\upsilon} - \mathbf{u}) - L(\boldsymbol{\upsilon} - \mathbf{u}) + \int_{\Sigma_C} f|\sigma_N(\mathbf{u})|(|\boldsymbol{\upsilon}_T - \mathbf{u}_T^t| - |\mathbf{u}_T - \mathbf{u}_T^t|) d\Sigma \geq 0, \quad \forall \boldsymbol{\upsilon} \in K, \quad \mathbf{u} \in K \tag{3.7}$$

where K is the set of kinematically admissible displacement fields for the instant of time $t + dt$.

The solution of inequality (3.7) is complicated by the fact that it contains the function $|\sigma_N(u)|$ non-differentiable at zero. This problem was first studied in detail by Duvaut (see Ref. 13), who proposed a method by which the quantity $|\sigma_N(u)|$ was replaced with a known function of the coordinates. This approach is justified by the fact that the shear forces in contact problems depend very weakly (and in some cases do not depend at all) on the contact pressure which can be determined from the solution of the problem without taking friction into account.

To take into account the dependence of the normal pressure on the friction forces, the following iterational process was proposed[3]:

$$a(\mathbf{u}^{(r+1)}, \boldsymbol{\upsilon} - \mathbf{u}^{(r+1)}) - L(\boldsymbol{\upsilon} - \mathbf{u}^{(r+1)}) + \int_{\Sigma_C} f|\sigma_N(\mathbf{u}^{(r)})|(|\boldsymbol{\upsilon}_T - \mathbf{u}_T^t| - |\mathbf{u}_T^{(r+1)} - \mathbf{u}_T^t|) d\Sigma \geq 0$$

$$\forall \boldsymbol{\upsilon} \in K, \quad \mathbf{u}^{(r+1)} \in K \tag{3.8}$$

where r is the number of the iteration. To implement the next iteration, we use the formula[2]

$$f|\sigma_N^{(r)}||\boldsymbol{\upsilon}_T - \mathbf{u}_T^t| = \max_{\boldsymbol{\mu}_T;\, |\boldsymbol{\mu}_T| \leq f|\sigma_N^{(r)}|} [\boldsymbol{\mu}_T(\boldsymbol{\upsilon}_T - \mathbf{u}_T^t)] \tag{3.9}$$

and we remove the constraint related to the non-penetration condition using the Lagrangian[2]; thereby, we transfer the problem of searching for the saddle point of the functional[14] – the minimum with respect to displacements and the maximum with respect to the contact interaction forces

The iterational method of searching for the saddle point is likewise described in previous studies. We will merelly only the following.

The convergence of the method at the loading step follows from the strict convexity of the functional with respect to displacements and concavity with respect to the contact interaction forces. The corresponding theorems were first proved by Kuhn and Tucker. The generalized formulations of these theorems as they apply to contact problems of mechanics in the Duvaut–Lions formulation can be found in Ref. 15. The problem of the convergence of the iteration process (3.8) was first touched upon in Ref. 3, where all transformations and estimates related to the problem of step minimization, rather than to the problem of searching for the saddle point; they are only valid on the assumption of the sufficient smoothness (double differentiability) of the solution. In the general case, this hypothesis concerning smoothness is not satisfied, although the principal constraint[3] imposed on the coefficient of friction can be substantiated by suitable regularization of the shape of the region and constraints.

Another approach to substantiating the method is based on the proof of the corresponding theorems for Lagrangians[16]. This method is implemented for friction contact problems in the Duvaut–Lions formulation (which occasionally is termed the generalized Signorini problem); numerical solutions were obtained for two-dimensional problems.

4. The solution of certain three-dimensional problems

Note that the latest software (ANSYS, NASTRAN, etc.) enables us to construct numerical solutions of three-dimensional problems for finite regions, albeit at the cost of considerable computer resources and special programming methods – parallel algorithms and multiprocessors. In order to concentrate on identifying the effects due to friction and unilateral constraints, we will use the boundary element method. The Boussinesq–Cerruti solutions will be used below. Strictly speaking, this version is only valid for a half-space (half-spaces). However, for small contact areas compared with the characteristic sizes of the contacting bodies, the method also gives acceptable solutions for regions of finite size.

We will assume that the region Ω is a linear elastic isotropic half-space. Let D be the maximum possible contact area, i.e., the area in which the contact interaction forces p(x) cannot equal zero. We will use a Cartesian system of coordinates $Oxyz$, the Oz axis of which will be directed deep into the half-space, while the Ox and Oy axes will be directed along the boundary of the half-space. Using the Boussinesq–Cerruti solution for single concentrated normal and shear forces on the boundary, we find the following relation between the displacements and forces at a point:

$$\mathbf{u}(\tilde{\mathbf{x}}) = \int_D \mathbf{L}(\tilde{\mathbf{x}}, \mathbf{x})[\mathbf{p}(\mathbf{x})] dD_x \tag{4.1}$$

where $\mathbf{L}[\mathbf{p}]$ is the kernel of the integral operator (4.1), $\tilde{x}$ is the point on the half-space boundary at which the displacements $\mathbf{u}$ are calculated and $\mathbf{x}$ is the point on the half-space boundary at which the vector of the density of surface (contact) forces $\mathbf{p}$ is specified. An explicit expression for $\mathbf{L}[\mathbf{p}]$ can be found, for example, in Ref. 17. We add constraints – inequalities (2.8)–(2.10) and (2.12) – to the integral equation (4.1), enabling us to find the contact area, the contact interaction force, and the boundary of the regions of adhesion and slip page in the contact area.

We will give a brief description of the principal iteration process at a fixed step with respect to t and for the first iteration.

1. We choose a certain zero approximation $\sigma_N^{(0)}, \sigma_T^{(0)}$ for the contact interaction forces.
2. We calculate the displacements $u^{(0)}$ in the calculated region D containing the maximum possible contact area; these displacements are found using of formula (4.1). Below, a square is used for region D, and the maximum possible contact area is determined from geometrical considerations (for example, for a sphere the maximum possible contact area is equal to the projection of the sphere onto the Oxy plane).
3. We correct the contact interaction forces in order to satisfy conditions (2.8)–(2.10) and (2.12); this is done in the following way

$$\sigma_N^{(1)} = P_N(\sigma_N^{(0)} + \rho_{0N}(\delta_N - u_N^{(0)})), \quad \boldsymbol{\sigma}_T^{(0)} = P_T(\boldsymbol{\sigma}_T^{(0)} + \rho_{0T}(\mathbf{u}_T^{(0)} - \mathbf{u}_T^t)) \tag{4.2}$$

Here

$$P_N(\sigma_N) = \begin{cases} \sigma_N, & \sigma_N \le 0, \\ 0, & \sigma_N > 0 \end{cases} \qquad P_T(\boldsymbol{\sigma}_T) = \begin{cases} \boldsymbol{\sigma}_T, & |\boldsymbol{\sigma}_T| \le f\left|\sigma_N^{(0)}\right| \\ \dfrac{\boldsymbol{\sigma}_T}{|\boldsymbol{\sigma}_T|} f\left|\sigma_N^{(0)}\right|, & |\boldsymbol{\sigma}_T| > f\left|\sigma_N^{(0)}\right| \end{cases}$$

are orthogonal projections of the corrected contact interaction forces onto the admissible set determined by the inequalities given above, ρ_{0N} and ρ_{0T} are number parameters controlling the convergence rate, and δ_N is a certain numerical measure of the gap between the half-space and the punch, for example, the value of the function Ψ at the corresponding point of the boundary.

5. Examples of numerical solutions

For the calculated region D we will choose a square whose centre coincides with the origin O of the system of coordinates Oxy and whose sides are parallel to the coordinate axes. We will divide the square into identical square elements and we approximate the contact interaction forces by constants in each element. We will number the nodes of the network by integers (k, l), $1 \le k, l \le (N+1)$, where N is the number of elements of division along each side of the square. We will number the boundary elements by integers (i, j), $1 \le i, j \le N$, where (i, j) is the number of the left-hand lower vertex of the element (the grid node number); the element itself will be denoted by T_{ij}. The displacements at node (i, j) will be combined into the column vector $\left\{U^{ij}\right\} = \left\{u_1^{ij}, u_2^{ij}, u_3^{ij}\right\}^T$ and the forces on the element T_{kl} will be combined into the column vector $\left\{P^{kl}\right\} = \left\{p_1^{kl}, p_2^{kl}, p_3^{kl}\right\}^T$. Then the displacements at node (k, l) will be related to the forces on the element T_{ij} by means of the matrix $\left[A_{ij}^{kl}\right]$, the structure of which follows from the structure of the operator (4.1):

$$[A_{ij}^{kl}] = \begin{Vmatrix} I_1 + I_2 & I_3 & I_4 \\ I_3 & I_1 + I_5 & I_6 \\ I_4 & I_6 & I_1 \end{Vmatrix}$$

where

$$\begin{aligned} I_1 &= 2(1-\nu)\int R^{-1} d\xi_1 d\xi_2, \quad I_2 = 2\nu\int R^{-3}(x_1-\xi_1)^2 d\xi_1 d\xi_2 \\ I_3 &= 2\nu\int R^{-3}(x_1-\xi_1)(x_2-\xi_2) d\xi_1 d\xi_2, \quad I_4 = (-1+2\nu)\int R^{-2}(x_1-\xi_1) d\xi_1 d\xi_2 \\ I_5 &= 2\nu\int R^{-3}(x_2-\xi_2)^2 d\xi_1 d\xi_2, \quad I_6 = (-1+2\nu)\int R^{-2}(x_2-\xi_2) d\xi_1 d\xi_2 \end{aligned} \tag{5.1}$$

The integrals $I_1, I_2, \ldots, I_6$ are evaluated for region T_{kl} and are in explicit form; there will be 36 different integrals in total if all possible symmetries of the kernel $\mathbf{L}$ are taken into account.

Using the superposition principle, we find the displacements at all nodes of the calculation grid

$$\{U^{ij}\} = \sum_{k,l} [A_{ij}^{kl}]\{P^{kl}\} \tag{5.2}$$

Thus, the calculation of the displacements in terms of the contact forces interaction reduces simply to multiplying a matrix by a vector. This enables us, on a computer with a comparatively small storage and operating speed, to achieve an extremely high accuracy of the results, for example to use grid sizes of the order of 200×200, which corresponds to an equivalent number of finite elements of $10^6 - 10^7$. Note also that we are actually dealing with a certain modification of the boundary element method – with maximum use of the available symmetries.

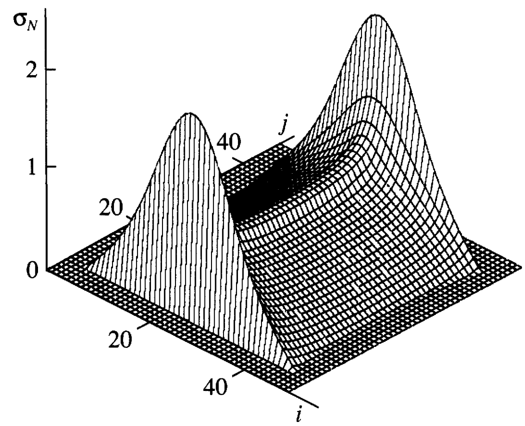

Fig. 1.

We will give a few examples of numerical solutions, arranged in order of increasing complexity of the punch geometry.

Example 1. We will consider, to begin with, the classic Hertzian problem of the indentation of a sphere into a half-space, treating it as a three-dimensional problem. This problem was solved in three formulations, in the first of which the classic hypotheses were used, in particular the hypothesis that the normal to the punch can be replaced by the normal to the half-space boundary. Furthermore, the problem was solved using both a one-step load procedure and a multistep load procedure; in all, 5–10 steps were used. Finally, a non-classical formulation was used, where the normal to the punch was used in the calculations, which naturally, even in problems ignoring friction, led to the appearance of horizontal components of the contact interaction forces.

Note that the advantage of multistep procedures is that they give a picture of the evolution of the areas of contact, adhesion and slippage, describe the distribution of tangential displacements more accurately and enable problems with complex loading to be considered (for example, initially to specify the penetration depth of the punch, and then its rotation).

An attempt was made to estimate the effect of frictional forces on the magnitude of the normal pressure. It turned out that this effect is very small, and the convergence of the iteration procedures is fairly slow, which requires a considerable amount of computer time.

In the calculations, the following input data were used: the length of the side of the square D on the half-space boundary was 2 (all parameters having a length dimension were reffered to the radius of the punch), the punch diameter was 2, the maximum penetration depth was 0.1; Young's modulus $E=5$ and Poisson's ratio $\nu = 0.23$. Note that no problems arose when using actual Young's modulus values, and the choice of $E=5$ was only for convenience in constructing the displacement stress graphs. The calculations were carried out on 50×50, 100×100 and 200×200 grids in order to assess the influence of boundary element size. The difference in the results on the graphs proved to be insignificant.

Example 2. Consider the problem of the penetration of a rigid cylinder of finite length into an elastic isotropic half-space. In this problem, all parameters with a length dimension were divided by the radius of the cylinder. The cylinder length was 1.8. The distribution of normal pressure and the modulus of the friction forces over the contact area is shown in Figs. 1 and 2 respectively. It can be seen that, on the edges of the cylinder, concentration of the contact stresses occurs, and here the peaks obtained grow as the number of elements increases. The physical interpretation of these peaks (theoretically equal to infinity) is normal: the solutions correspond to the rounded edges of the cylinder, the radius of curvature of which is equal to the characteristic size of a boundary element.

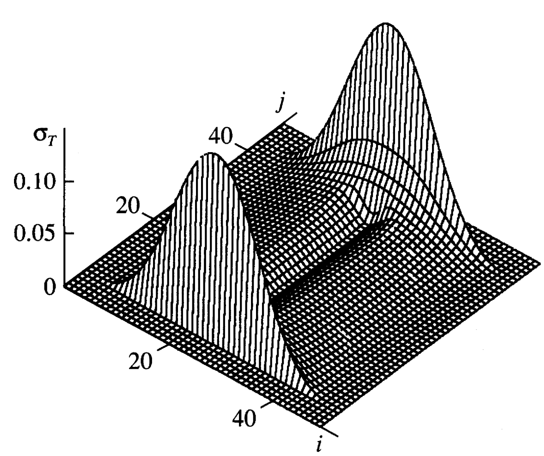

Fig. 2.

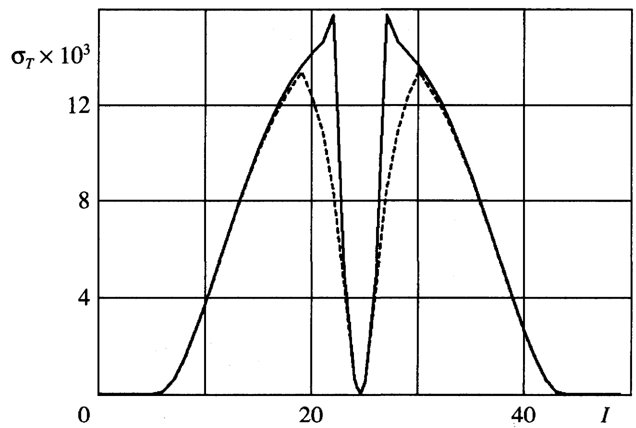

Fig. 3.

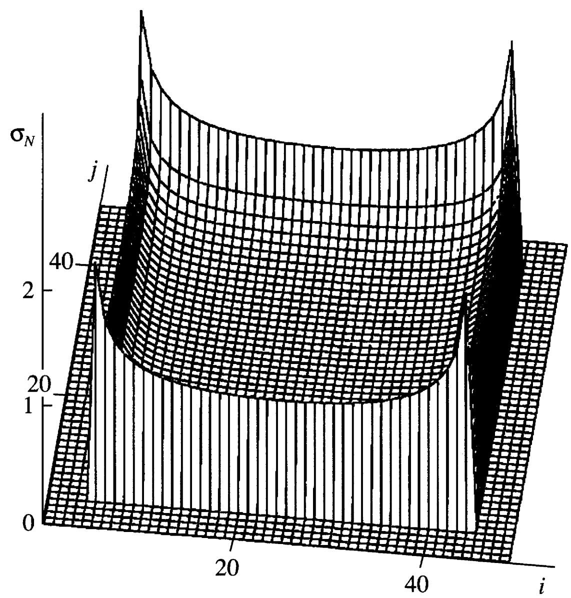

Fig. 4.

Note that the surface section corresponding to the distribution of the friction forces in a plane perpendicular to the cylinder axis is a curve that has a maximum on the boundary of the regions of adhesion and slip-page. The modulus of the friction forces in an average section of the cylinder is shown in Fig. 3.; I is the number of the element in the cross-section, $|\sigma_T|$ is the modulus of the vector of the friction forces, the continuous curve corresponds to a single loading step and the dashed curve corresponds to ten steps.

The maxima on the curves depend on the number of loading steps. Stabilization begins roughly at ten steps for a penetration depth of 0.1. The effect revealed earlier in the plane problem is confirmed: for a single loading step (which corresponds to replacement of the velocities in the Amonton–Coulomb friction law by displacements), the maximum of the friction forces comprises an irregular point on the curve obtained – the first derivative at this point has a break of the first kind; when the multistep load procedure is used, the curve becomes smooth, and the maximum value of the friction forces is reduced.

We will now consider now the solution of the problem of the penetration of a cube into a linear elastic isotropic half-space for a cube of side 1.6. As in the two previous problems, the number of boundary elements is 50×50, the maximum penetration depth is 0.1, the number of load steps is 10, and the coefficient of friction is 0.2. We will present the most interesting of the results obtained. Figs. 4 and 5 show the normal pressure distribution and the modulus of the friction forces respectively. Note that, as the number of elements increase, the maxima of the contact forces in the corners and on the ridges of the parallelepiped increase (as in the previous problem, theoretically they are equal to infinity); as before, the numerical solutions are interpreted as the solutions for a punch with rounded edges with a radius of curvature equal to the element size. It can also be seen that the singularity at the corners is stronger than on the ridges.

6. Possible artefacts

When debugging the algorithms for the numerical solution of friction contact problems, it turned out that artefacts (i.e., mechanical effects not occurring in real systems) can result from an inappropriate choice of the initial load-step approximation when there is more than one loading step.

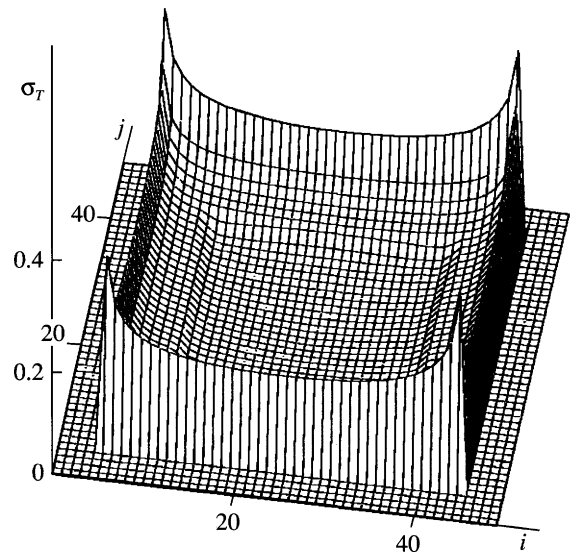

Fig. 5.

We will give an example of such an effect that arises when solving the two-dimensional problem of the penetration of a rigid, round, rough punch into an elastic region in the form of an *ABCD* rectangle. The penetration depth increases monotonically, and the areas of contact, adhesion and slippage increase along with it.

Fig. 6 shows the distributions of the friction forces as a function of the penetration depth. The maximum penetration depth is equal to 0.002*BC*, the number of steps is 1 or 2, and the coefficient of friction $f = 0.4$. The figure shows the left-hand halves of the curves; the right-hand halves are obtained by antisymmetric reflection about to the point $I = 100$, where I is the number of the point of subdivision of the top side of the rectangle into segments of identical length (boundary elements). The size of the rectangle is 2×2, and the number of boundary elements is 500 (50 elements each on sides *AB* and *CD*; 200 elements each on sides *BC* and *DA*; area of possible contact area – side *BC*). The punch radius R was chosen to be 100*BC* in order for the effect revealed to be as marked as possible – as in the ratio R/BC decreases, the artefact weakens. Curve *1* corresponds to the first loading step, equal to 0.001, and curves 2–4 correspond to the penetration depth, equal to 0.002. Curve 3 was obtained for a single loading step, and curve 2 was obtained using a zero approximation of the solution constructed at the first step; in plotting curve 4, the normal pressure distribution, plotted for the actual value of the penetration depth and zero friction forces, was used as the zero approximation.

The artefact in question – the "tooth" in Fig. 2 – does not disappear when the number of penetration steps is increased; calculations were carried out for 10–15 steps. As the number of iterations increases, the artefact slowly disappears, but the convergence is so slow that the power of the computer is insufficient to arrive at a solution in an acceptable time.

Numerical analysis showed that the appearance of a 'tooth' on curve *2* is related to the fact that the magnitude of the friction forces depends on the magnitude of the normal pressure, which in the zero approximation is equal to zero in the region to which the contact area proagates, and with the fact that, in iterations, the friction forces are corrected by changing the tangential displacements, that are close to zero in the area mentioned (see formula (4.2)). This fact implies non-uniform convergence of successive approximations to the solution. This non-uniformity, it seems, can be eliminated by using variable values of the parameter ρ_{0T} in formula (4.2). However, it is simpler to change the choice of load-step zero approximation, as was subsequently done. Note that this artefact (non-uniform convergence) also occurs in three-dimensional problems.

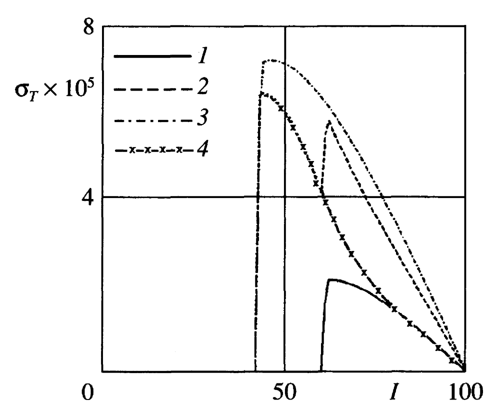

Fig. 6.

Acknowledgements

This research was in partly financed by the Russian Foundation for Basic Research (07-08-0269a, 07-01-13523 and NSh-1245.2006.1).

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Translated by P.S.C.